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Highest weights, projective geometry, and the classical limit II. Coherent states and integrable systems

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Abstract

Coherent states of the commutation relations, like highest weight vectors for compact semi-simple Lie groups, satisfy quadratic equations. This paper explores the situation for quadratic varieties of vectors in some other infinite-dimensional representations, the tau functions for loop groups providing one example. Other generalisations are discussed.

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1. Introduction

In the first of this pair of papers on highest weights [6], we investigated some geometrical features of the characterisation of highest weights presented in [5]. In this sequel, we shall consider some new examples for locally compact and infinite-dimensional groups, and explain how the geometry is related to the classical limit.

We shall generally use the same conventions as in [6], to which we refer for any unexplained notation. We shall, in particular, use the following generalised notion of dominant vectors introduced there.

Definition. For any group G and G-module V let us call a vector $v \in V$ *dominant* if $v \otimes v$ generates an irreducible submodule of the tensor product representation or, more generally, a submodule of minimal composition length.

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Although [6] introduced some generalisations involving higher tensor powers in this paper we shall use only dominant vectors satisfying quadratic constraints.

The paper is organised as follows. It is known that the solutions of many integrable systems such as the KP hierarchy can be identified with the orbits of highest weight vectors for appropriate loop groups, and that these can be characterised by the quadratic equations known as the Hirota bilinear form, [7,15]. From the above perspective this involves an extension into infinite dimensions of a result proved in [5] classifying dominant vectors for the Heisenberg group as coherent states. This can be done by combining characterisations of dominant vectors for infinite-dimensional vector groups, considered in Section 2, and for certain induced representations of locally compact semi-direct product groups discussed first in Section 3 and then applied in Sections 4 and 5 to the $ax + b$ group and to Mackey's generalised Heisenberg groups. The application to the Hirota bilinear form is then briefly discussed in Section 6. There also seems to be a relationship with the notion of quasi-free states in quantum field theory, both for bosonic and for fermionic systems.

Section 7 explains how one might extend the idea of a dominant vector to generalised function spaces.

2. Coherent states

In general it is difficult to characterise the vectors which satisfy quadratic equations, particularly in an infinite-dimensional representation. However, there are some cases, such as the Heisenberg group, discussed as the final example in [5], in which this can be done. In this section, we generalise that example.

Theorem 2.1. *Let V be a real vector group (possibly infinite-dimensional) with a symplectic form s and the totally skew multiplier $\sigma = \exp(\frac{1}{2}\mathrm{i}s)$, and let $\mathrm{CCR}(V)$ be the C^*-algebra generated by unitary elements $\delta(x)$, with $x \in V$, satisfying*

$$\delta(x + y) = \sigma(x, y)\delta(x)\delta(y).$$

Let W be an irreducible $*$-representation of $\mathrm{CCR}(V)$ on a Hilbert space $\mathcal{H}$ which takes $\delta(x)$ to $W(x)$, which is regular in the sense that the map $\lambda \in \mathbf{R} \mapsto W(\lambda x)$ is strongly continuous for all $x \in V$. A vector $v \in \mathcal{H}$ is dominant if and only if the σ-positive function $\omega(x) = \langle v, W(x)v\rangle$ has the form

$$\omega(x) = \exp(-\tfrac{1}{4}B(x, x) + \mathrm{i}\lambda(x)),$$

where the linear functional λ and the bilinear form $B(x, y)$ are real and the latter is positive definite and satisfies

$$1 = \sup\{s(x, y)^2/B(x, x)B(y, y) : x, y \in V\backslash\{0\}\}.$$

Proof. The vector v and representation W can be recovered from the σ-positive function $\omega(x)$ by the GNS construction. Now $W \otimes W$ defines the irreducible $\sigma \times \sigma$-representation of $V \oplus V$, and $v \otimes v$ defines the σ-positive function $\omega_2(x, y) = \langle v \otimes v, W(x)v \otimes W(y)v\rangle$

on $V \oplus V$. The rotation of $V \oplus V$ which sends (x, y) to $((x-y)/\sqrt{2}, (x+y)/\sqrt{2})$ preserves the class of the multiplier, and so defines an automorphism α of $\mathrm{CCR}(V \oplus V)$. We then have $\omega(x)\omega(y) = \omega_2(x, y) = \omega_2(\alpha((x+y)/\sqrt{2}, (y-x)/\sqrt{2}))$. When V is finite dimensional Mackey's generalisation of the Stone–von Neumann theorem tells us that W and $W \otimes W$ are, up to equivalence, unique, so that $W \otimes W$ and the GNS representation determined by $\omega_2 \circ \alpha$ are equivalent, but in infinite dimensions this is usually false. The vector $v \otimes v$ generates an irreducible representation if and only if the $\sigma \times \sigma$-positive function $\omega(x)^2 = \omega_2(\alpha(\sqrt{2}x, 0))$ defines a pure state, that is, it is extreme (see [6, Section 8]). Let v_α be the GNS vector defined for the $\sigma \times \sigma$-representation W_α defined by $\omega_2(\alpha(x, y))$. The map taking $W_\alpha(x, 0)v_\alpha \otimes W_\alpha(0, y)v_\alpha$ to $W_\alpha(x, y)v_\alpha$ is well defined, and can be extended to a map from the tensor product of the Hilbert spaces spanned by $W_\alpha(V, 0)v_\alpha$ and by $W_\alpha(0, V)v_\alpha$ to the space generated by v_α under $W_\alpha(V \oplus V)$. The first and third of these spaces are known to be irreducible, and that forces the second to be irreducible too. But then T is a non-zero intertwining operator between irreducible representations and so is an isomorphism. As a result v_α can be expressed as a tensor product of vectors for $W_\alpha(V, 0)$ and $W_\alpha(0, V)$ and so $\omega_2 \circ \alpha$ must factorise, in other words for some ϕ and ψ we must have

$$\omega_2(\alpha(x, y)) = \phi(\sqrt{2}x)\psi(\sqrt{2}y),$$

or equivalently

$$\omega(x)\omega(y) = \phi(x+y)\psi(y-x)$$

for all x and y in V.

As the σ-positive definite function associated to a representation, ω is continuous on one-parameter subgroups. We concentrate our attention on the case of x and y in $\mathbf{R}$, satisfying the functional equation. Since $\omega(0) = \|v\|^2$ is positive there is some neighbourhood of 0 in which ω does not vanish. Further, if ω vanishes at all, then it must be possible to find a such that $\omega(2a) = 0$, but neither $\omega(a)$ nor $\omega(-a)$ vanishes. However, by using the functional identity twice and regrouping the ϕ and ψ, we see that

$$\begin{aligned}\omega(x+y+z)\omega(x)\omega(y)\omega(z) &= \phi(2x+y+z)\psi(y+z)\phi(y+z)\psi(y-z)\\ &= \omega(y+z)\omega(0)\omega(x+y)\omega(x+z).\end{aligned}$$

Setting $x = y = a = -z$ gives $\omega(a)^3\omega(-a) = \omega(2a)\omega(0)^3$, which would lead to a contradiction. Since this works for all one-parameter subgroups we conclude that ω has no zeros.

We may now rearrange the previous identity as

$$\frac{\omega(x+y+z)\omega(0)}{\omega(x)\omega(y+z)} = \frac{\omega(x+y)\omega(0)}{\omega(x)\omega(y)}\,\frac{\omega(x+z)\omega(0)}{\omega(x)\omega(z)},$$

which means that $\beta(x, y) = \omega(x+y)\omega(0)/\omega(x)\omega(y)$ is a symmetric bicharacter on V. However, every symmetric bicharacter on a vector group is the exponential of a bilinear form, so for some such form B we have $\beta(x, y) = \exp(-\frac{1}{2}B(x, y))$. It is now easy to check that $\omega(x)\exp(\frac{1}{4}B(x,x))/\omega(0)$ is a character of V. It can therefore be written as $\exp(\mathrm{i}\lambda(x))$ for some linear functional λ. Thus we have

$$\omega(x) = \exp(-\tfrac{1}{4}B(x,x) + \mathrm{i}\lambda(x)).$$

One interesting consequence of this approach in finite dimensions is that ω is a smooth function so that v is automatically a C^∞ vector.

For positivity $\omega(-x) = \overline{\omega(x)}$ so that B and λ must be real. Moreover, $|\omega(x)|^2 \le 1$ so that the quadratic form $B(x,x)$ must be positive definite, and from the three-dimensional constraints on positivity, it must also satisfy

$$B(x,x)B(y,y) \ge |B(x,y) + \mathrm{i}s(x,y)|^2 \ge s(x,y)^2$$

for all x and y in V (this is then sufficient). The irreducibility of W imposes the further constraint that the inequality $B(x,x)B(y,y) \ge s(x,y)^2$ must be sharp. □

For finite degrees of freedom this recovers the result already given in [5], where it was shown that the dominant vectors could be identified with the coherent states. We shall use an infinite-dimensional version later when we consider the Hirota bilinear form for loop groups.

We readily calculate that

$$\langle W(z)v, W(x)W(z)v\rangle = \frac{\sigma(x,z)}{\sigma(z,x)}\langle v, W(x)v\rangle = \exp(-\tfrac{1}{4}B(x,x) + \mathrm{i}\lambda(x) + \mathrm{i}s(x,z)),$$

so that in finite dimensions we can remove the linear term $\lambda(x)$ by appropriate choice of z. (In infinite dimensions this is not always possible.) The quadratic term is, however, the same for all the vectors on the orbit of v. We note that

$$\phi(x) = \omega(\tfrac{1}{2}x)^2 = \exp(-\tfrac{1}{8}B(x,x) + \lambda(x)),$$

and

$$\psi(y) = \omega(\tfrac{1}{2}y)\omega(-\tfrac{1}{2}y) = \exp(-\tfrac{1}{8}B(y,y)).$$

In particular ψ is the same for all vectors on the orbit of v, though it varies from one orbit to another.

It is worth remarking that one can consider analogues of Theorem 3.1 in [6], classifying the linear transformations which preserve the dominant vectors. One may argue similarly using the holomorphic function representation of the representation space. For example, for $V = \mathbf{R}^2$ the dominant vectors form various copies of $\mathbf{C}$ each with its own complex structure. Now, the biholomorphic mappings of $\mathbf{C}$ are the linear transformations, which correspond exactly to the complexification of the oscillator group, $U(1).\mathbf{R}^2$. In general one obtains $U(n).\mathbf{R}^{2n}$. Any change of complex structure can be implemented by an element of the symplectic group, and then one gets the complexification of the affine symplectic group.

We also remark that, just as for the Clifford algebras, the dominant vectors define quasi-free states in the sense of quantum field theory [2], that is, states determined by their values on elements of the algebra which are at most quadratic in the generators, so that this is a feature of both bosonic and fermionic systems.

3. Dominant vectors for some induced representations

Dominant vectors can also be characterised for induced representations of certain semi-direct product groups (see [10] and references therein).

Theorem 3.1. *Let G be a separable locally compact group which is the semi-direct product of abelian groups $H.N$, with H, connected, having a regular action on the dual $\hat{N}$. Let $\nu \in \hat{N}$ have a trivial stabiliser in H, so that the induced representation $U = \mathrm{ind}_N^G \nu$ is irreducible. Assume further that ν^2 also has a trivial stabiliser in H and that $(H.\nu)^2 \subseteq H.\nu^2$. Dominant vectors exist if and only if for some function b on $\nu^{-1}(H.\nu)$,*

$$b(\nu^{-1}h_1.\nu)b(\nu^{-1}h_2.\nu)b(\nu^{-1}h_1h_2.\nu)^{-1}$$

defines a bicharacter, in which case the most general dominant vector v has the form

$$v(h.\nu) = c.b(\nu^{-1}(h.\nu))\chi(h)$$

for some constant c and character $\chi \in \hat{H}$.

Proof. The induced representation U can be realised on the functions $\psi \in L^2(H.\nu)$ (with the invariant measure on $H.\nu \subseteq \hat{N}$) by

$$(U(h,n)\psi)(k.\nu) = (h^{-1}k.\nu)(n)\psi(h^{-1}k.\nu).$$

To analyse the action on the tensor product we note that $h.\nu$ determines $h \in H$ uniquely, so that we may define an operator T from $L^2(H.\nu \times H.\nu)$ to $L^2((H.\nu)^2, H)$ by

$$(T\Psi)(h_1.\nu, h_2.\nu) = \Psi((h_1.\nu)(h_2.\nu), h_1^{-1}h_2).$$

Moreover, $(h_1.\nu)(h_2.\nu) = h_1.(\nu(h_1^{-1}h_2.\nu))$ can be written as $k.\nu^2$, and then h_1 determined from the identity $h_1^{-1}k.\nu^2 = (\nu(h_1^{-1}h_2.\nu))$, so T is invertible. We then have

$$\begin{aligned}
&(T^{-1}U(h,n) \otimes U(h,n)T\Psi)((h_1.\nu)(h_2.\nu), h_1^{-1}h_2)\\
&= (U(h,n) \otimes U(h,n)T\Psi)(h_1.\nu, h_2.\nu)\\
&= (h^{-1}h_1.\nu)(n)(h^{-1}h_2.\nu)(n)T\Psi(h^{-1}h_1.\nu, h^{-1}h_2.\nu)\\
&= h^{-1}((h_1.\nu)(h_2.\nu))(n)\Psi(h^{-1}((h_1.\nu)(h_2.\nu)), h_1^{-1}h_2).
\end{aligned}$$

Now the first argument of Ψ lies in $(H.\nu)^2 = H.\nu^2$ so that this can be reinterpreted as saying that

$$(U \otimes U)T = T(U_2 \otimes 1),$$

where $U_2 = \mathrm{ind}_N^G \nu^2$, which is also irreducible. The vector $v \otimes v$ can therefore only generate an irreducible subrepresentation if it is of the form $T(u \otimes w)$ for some vectors u and w. More precisely, we have

$$v(h_1.\nu)v(h_2.\nu) = u((h_1.\nu)(h_2.\nu))w(h_1^{-1}h_2).$$

If we take $h_2 = hh_1$ and integrate this identity over h_1, we obtain

$$\langle \bar{v}, U(h)v\rangle = \left(\int u(h_1)\,\mathrm{d}h_1\right) w(h),$$

from which it follows that w is continuous. This in turn enables us to show that w vanishes identically if it vanishes at all (as in the last section). We are clearly not interested in the trivial case when $w = 0$, so we may assume that w never vanishes, and then u never vanishes either. We now use the functional equation linking v to u and w to show that

$$\begin{aligned}\frac{v(h_1.\nu)v(h_2.\nu)v(h_3.\nu)v(h_1h.\nu)}{v(h_1h_2.\nu)v(h_1h_3.\nu)v(\nu)v(h.\nu)} &= \frac{u((h_1.\nu)(h_1h.\nu))w(h)u((h_2.\nu)(h_3.\nu))w(h_2^{-1}h_3)}{u((h_1h_2.\nu)(h_1h_3.\nu))w(h_2^{-1}h_3)u(\nu(h.\nu))w(h)}\\ &= \frac{u(h_1.(\nu(h.\nu)))u((h_2.\nu)(h_3.\nu))}{u(h_1((h_2.\nu)(h_3.\nu)))u(\nu(h.\nu))}.\end{aligned}$$

If it is possible to choose $h \in H$ such that $\nu(h.\nu) = (h_2.\nu)(h_3.\nu)$, then this collapses down to 1, giving

$$\frac{v(h_1.\nu)v(h_2.\nu)}{v(h_1h_2.\nu)v(\nu)}\,\frac{v(h_1.\nu)v(h_3.\nu)}{v(h_1h_3.\nu)v(\nu)} = \frac{v(h_1.\nu)v(\nu^{-1}(h_2.\nu)(h_3.\nu))}{v(\nu^{-1}(h_1h_2.\nu)(h_1h_3.\nu))v(\nu)},$$

showing that when $b_\nu(\nu^{-1}(h.\nu)) = v(h.\nu)/v(\nu)$,

$$(\nu^{-1}(h_1.\nu), \nu^{-1}(h_2.\nu)) \mapsto \frac{b_\nu(\nu^{-1}(h_1.\nu))b_\nu(\nu^{-1}(h_2.\nu))}{b_\nu(\nu^{-1}h_1h_2.\nu)}$$

defines a bicharacter. If no such b_ν exists, then there can be no v for which $v \otimes v$ generates an irreducible. When such b_ν does exist, then the ratio $c = b_\nu/b$ of two of them satisfies

$$c(\nu^{-1}(h_1.\nu))c(\nu^{-1}(h_2.\nu)) = c(\nu^{-1}h_1h_2.\nu),$$

so that $\chi(h) = c(\nu^{-1}(h.\nu))$ is a character on H, and $v(h.\nu) = v(\nu)b(\nu^{-1}(h.\nu))\chi(h)$ has the required form. □

It is also worth mentioning another property of semi-direct product groups $H.N$. Suppose that v is a dominant vector for an irreducible representation of N and the restriction of $U \otimes U$ to the cyclic subspace generated by $v \otimes v$ is the irreducible V. Suppose also that H_U and H_V are the subgroups of H stabilising the equivalence classes of the representations U and V. Then v is also a dominant vector for $(H_U \cap H_V).N$. This follows more or less immediately from the definitions. In some cases such as the semi-direct product of the metaplectic group with the group of the canonical commutation relations, and the semi-direct product of the diffeomorphism group, $\mathrm{Diff}(S^1)$, with appropriate central extensions of a loop group, it is known that $H_U \cap H_V = H$, so that the dominant vectors for the normal subgroup are still dominant for the whole semi-direct product.

Induced representations can also appear in a rather different guise. Any square-integrable representation of a group G can be embedded in the left-regular representation on $L^2(G)$ by a map T_v as described in Section 1. The tensor product representation can be similarly imbedded in $L^2(G \times G)$. Now the map taking $\Phi \in L^2(G \times G)$ to $S\Phi(x, y) = \Phi(x, xy)$

intertwines the tensor product representation of G and the left regular representation on the first argument alone for

$$(SU(g)\otimes U(g)S^{-1}\Phi)(x,y) = (U(g)\otimes U(g)S^{-1}\Phi)(x,xy)$$
$$= (S^{-1}\Phi)(g^{-1}x, g^{-1}xy) = \Phi(g^{-1}x, y).$$

Thus $S\Phi$ generates an irreducible representation under $U\otimes U$ if and only if Φ generates an irreducible representation under $U\otimes 1$, and this could happen if, for example, $\Phi = u\otimes w$ where u generates an irreducible subrepresentation of the left-regular representation. So if $S\Phi = u\otimes w$ and $\Phi = v\otimes v$, then the vector v is dominant. We may as well assume that the function $v(1)\neq 0$ since, for instance, $T_v v(1) = \langle v, v\rangle$.

Theorem 3.2. *If $S(v\otimes v) = u\otimes w$, where $v(1)\neq 0$, then v is supported on a subgroup H, and for some linear character $\chi\in\hat{H}$, the restriction of v to H is given by $v(g) = v(1)\chi(g)$.*

Proof. By assumption we have $v(x)v(xy) = u(x)w(y)$. Taking $x = 1$, we see that $v(1)v(y) = u(1)w(y)$, whilst when $y = 1$ we have $v(x)^2 = u(x)w(1)$. Substituting these back into the original equation gives

$$u(1)w(1)v(x)v(xy) = w(1)u(x)u(1)w(y) = v(1)v(x)^2 v(y).$$

Doing some obvious cancellation, and using $v(1)^2 = u(1)w(1)$, we arrive at

$$v(1)v(xy) = v(x)v(y),$$

provided that $v(x)\neq 0$. From this equation, it is clear that if neither $v(x)$ nor $v(y)$ vanishes, then $v(xy)$ is also non-zero. Moreover, setting $xy = 1$ we see that $v(x^{-1})\neq 0$, so that v is supported on a subgroup H, and there we immediately see that $v/v(1)$ is a linear character. □

Although this provides a sufficient condition for the dominance of v when u generates an irreducible representation, it is certainly not necessary. In fact the result is really most useful for finite groups, since it requires that functions supported on subgroups be square-integrable. The function v is an element of the induced representation subspace $\mathrm{ind}_{\mathrm{H}}^{G}\chi$.

4. Affine group in one dimension

As a first application of Theorem 10.1 we take G to be the affine linear group AGL(1, $\mathbf{R}$) which is the semi-direct product of the positive reals $H = \mathbf{R}^+$ and $N = \mathbf{R}$, with the natural multiplication action of H on N, also commonly known as the $ax + b$ group. We take the character $\nu(a) = \exp(ipa)$ of N, which has trivial stabiliser when $p\neq 0$. For $\alpha\in H$ we readily check that

$$\nu^{-1}(\alpha.\nu)(a) = \mathrm{e}^{\mathrm{i}(\alpha-1)\mathrm{pa}}.$$

Two such elements defined by α and γ are composable just when $\alpha+\gamma > 1$. The subset of elements for which $\alpha > 1$ defines a semi-group isomorphic to $\mathbf{R}^+$ in $\hat{N}\cong\mathbf{R}$. We readily calculate that $(\alpha.\nu)(\gamma.\nu) = (\frac{1}{2}(\alpha+\gamma).\nu)^2$, so that $(H.\nu)^2$ is contained in $H.\nu^2$. For α and γ greater than 1, the bicharacters on the semi-group $\mathbf{R}^+$ are, as above, of the form

$$\mathrm{e}^{A(\alpha-1)(\gamma-1)} = \mathrm{e}^{-A(\alpha-1)}\,\mathrm{e}^{A(\gamma-1)}\,\mathrm{e}^{A(\alpha\gamma-1)},$$

so that we may take $b((\alpha-1)p) = \exp(-A(\alpha-1))$. The general character on H has the form $\alpha \mapsto \alpha^r$, for $r \in \mathbf{C}$, so that, by changing the constant, the theorem gives

$$v_{Ar}(\xi.\nu) = c\xi^r\,\mathrm{e}^{-A\xi}$$

as the most general vector v for which $v \otimes v$ generates an irreducible representation.

Since

$$(U(\alpha,a)v_{Ar})(\xi.\nu) = c\,\mathrm{e}^{\mathrm{i}pa\xi}\alpha^r\xi^r\,\mathrm{e}^{-A\alpha\xi} = \alpha^r v_{\alpha A-ipa,r}(\xi),$$

we may use a to remove the imaginary part of A and then re-scale using α, showing that every v_{Ar} lies on the orbit of v_{1r}. However, we still have a one-parameter family of such vectors so that unlike many of the other examples considered the automorphism group does not act transitively on the dominant vectors. (Every outer automorphism of G is equivalent to that sending (α, a) to $(\alpha, -a)$. This interchanges the representations induced using p and $-p$, so there is no corresponding action on dominant vectors.)

One readily calculates the associated positive definite function to be

$$\omega_{Ar}(\alpha,a) = \frac{\langle v_{Ar}, U(\alpha,a)v_{Ar}\rangle}{\|v_{Ar}\|^2} = \alpha^r\left(\frac{\overline{A}+A}{\overline{A}+\alpha A - \mathrm{i}pa}\right)^{(r+\overline{r})}.$$

which as a mesomorphic function of $A\alpha - \mathrm{i}pa$, reveals a hidden complex structure on G.

By inspection of our earlier formulae we see that the vector v_{Ar} is not an eigenvector for any non-trivial group elements, but it is an eigenvector for some elements of the complex Lie algebra. The Lie algebra is spanned by the elements P and Q which are the infinitesimal generators of scale changes and translations, respectively. In the representation U these are represented by $\xi(\mathrm{d}/\mathrm{d}\xi)$ and multiplication by $\mathrm{i}p\xi$. We therefore have

$$(pP - \mathrm{i}AQ)v_{Ar} = p\xi\left(\frac{\mathrm{d}}{\mathrm{d}\xi} + A\right)(c\xi^r\,\mathrm{e}^{-A\xi}) = rpv_{Ar}.$$

This means that v_{Ar} is a coherent state in the sense of Perelomov, [12,13, Section 2.4]. This suggests that one could regard the dominant vectors as defining coherent states. There are, however, other generalisations, such as those presented in [1], which do not immediately fit with this picture.

5. Mackey–Heisenberg groups

For any abelian group H we may consider Mackey's generalisation of the Heisenberg group [9,10], $H \times \hat{H} \times \mathbf{T}$ with multiplication

$$(\xi, x, X)(\eta, y, Y) = (\xi\eta, xy, \xi(y)^{-1}XY).$$

We can identify this as the semi-direct product of H and the abelian normal subgroup $N = \hat{H} \times \mathbf{T}$, in which $\xi.(y, Y) = (y, \xi(y)^{-1}Y)$. We take the character $\nu(y, Y) = Y$ and note that $(\xi.\nu)(y, Y) = \nu(\xi^{-1}.(y, Y)) = \xi(y)Y$. The stabilisers of ν and ν^2 are trivial and, moreover, $\nu^{-1}\xi.\nu$ is just the lift of ξ from $\hat{H}$ to N. This immediately shows that $(\xi.\nu)(\eta.\nu) = \nu(\xi\eta.\nu)$. If H is 2-divisible, then we may rewrite this as $((\xi\eta)^{1/2}.\nu)^2$, so that $(H.\nu)^2 \subseteq H.\nu^2$.

We now consider the bicharacters on $\nu^{-1}(H.\nu) \sim H$, and first the case of $H = \mathbf{R}^n$. The bicharacters on $\mathbf{R}^n$ all take the form

$$\beta(\xi, \eta) = \exp(-A(\xi, \eta))$$

for some symmetric bilinear form A. We may satisfy the hypothesis of the theorem by taking $b(\xi) = \exp(-\frac{1}{2}A(\xi, \xi))$. Bearing in mind the fact that the most general character χ of $H = \mathbf{R}^n$ has the form $\chi(\xi) = \exp(ia.\xi)$ for some $a \in \mathbf{C}^n$, we deduce that $v(\xi.\nu)$ is the exponential of a quadratic in ξ. This recovers the coherent sates in finite dimensions by yet another argument.

Unfortunately this argument may founder when H or $\hat{H}$ is discrete. For example $H = \mathbf{Z}$ is not 2-divisible, whilst when $H = \mathbf{T} = \hat{\mathbf{Z}}$ and the stabiliser of ν is trivial, that of ν^2 is ± 1. Nonetheless it is possible to rescue the results by making a minor modification of the previous arguments when H is discrete. We note that the operator T from $L^2(H \times H)$ to a subspace of itself defined by

$$(T\Psi)(\xi, \eta) = \Psi(\xi\eta, \xi\eta^{-1})$$

is invertible on the functions supported on pairs $(x, y) \in H \times H$ for which xy and xy^{-1} are both squares, since then we have

$$(T\Phi)(x, y) = \Phi((xy)^{1/2}, (xy^{-1})^{1/2}).$$

This trick relies on being able to restrict our attention to functions supported on the subset giving squares, and fails, for example, for the Euclidean group $\mathbf{SO}(2).\mathbf{R}^2$. We now have

$$(T^{-1}U(\alpha, a, A) \otimes U(\alpha, a, A)T\Psi)(x, y) = A^2\alpha(xa^{-2})\Psi(xa^{-2}, y)$$

and the argument proceeds as before. In the case of the group $\mathbf{Z}^n \times \mathbf{T}^n \times \mathbf{T}$ $(H = \mathbf{Z}^n)$ we find dominant vectors

$$v(m) = c\exp(\mathrm{i}\lambda(m))\exp(-\tfrac{1}{2}B(m, m)),$$

where $m \in \mathbf{Z}^n$, λ is a complex-valued linear functional restricted to $\mathbf{Z}^n$ and B is the restriction of a symmetric bilinear form with positive real part. This is the Fourier transform of a theta function, and the corresponding positive function ω is also a theta function. Since H is not connected these are not the only possibilities, and here are also some limiting cases as eigenvalues of B tend to infinity. If suitably linked to λ this can give us δ functions in m, both in v and ω.

Theta functions can also appear in a slightly different way as positive-definite functions for the projective representations of vector groups. This occurs, for example, in the case of the representations of loop groups defined by Riemann surfaces [3]. In the simplest case we take a Riemann surface Σ with one boundary circle and form its Schottky double $\tilde{\Sigma}$ which has even genus g, and the surface $\hat{\Sigma}$ obtained by adding a single cap. The vector group $G = H^1(\hat{\Sigma}) \sim \mathbf{R}^g$ has a natural symplectic form which exponentiates to a multiplier σ. As in [3] there is a σ-positive-definite function for the vector group defined by a theta function θ on $\tilde{\Sigma}$. Now θ^2 inherits from θ equivariance properties under translation by elements of the lattice $\Gamma = H^1(\hat{\Sigma}, \mathbf{Z})$, but the relevant multiplier is now σ^2, and so this lattice is no longer self-reciprocal. In fact, writing $\Gamma^{(2)} = \{x \in G : \sigma(x, y)^2 = \sigma(y, x)^2\}$ we calculate that $\Gamma^{(2)}/\Gamma = \mathbf{Z}_2^g$. From this we can show that θ^2 defines a direct sum of $2^{(1/2)g}$ irreducibles [4,10]. Thus the corresponding vector v fails to be dominant in a very well-defined way.

6. Hirota bilinear form

A useful example of quadratic equations in infinite dimensions occurs for the case of the Hirota bilinear form for the KP and other hierarchies of integrable equations. It is known that the solutions of these equations can be obtained from τ functions which can be constructed from highest weight vectors v for the basic representation of an affine Lie algebra or loop group [7,14,15], and the references therein. The Hirota bilinear form provides quadratic equations for the τ functions and can be regarded as giving quadratic equations satisfied by vectors on the orbit of v [7, Proposition 14.11] (for a simpler form of the equation see [8, Remark 4.11]). Conversely one can use the results of the preceding sections to characterise all vectors satisfying these quadratics.

To illustrate the idea we consider only the group of smooth loops from the circle to the torus, $G = \mathrm{Map}(S^1, \mathbf{T})$ with the standard basic multiplier

$$\sigma(e^{if}, e^{ig}) = \exp\left(-\frac{i}{4\pi}\left(\int_0^{2\pi} g\,df + f(0)(g(2\pi) - g(0))\right)\right).$$

This loop group can be expressed as a direct product $\mathbf{Z} \times \mathbf{T} \times V$, where the first two factors give the winding number terms and the constants and V is the infinite-dimensional vector group of real-valued functions on S^1 whose mean vanishes, [14]. By [6 Theorem 4.1] the irreducible σ-representations for G are tensor products of irreducible projective representations of $\mathbf{Z} \times \mathbf{T}$ and V, and the dominant vectors v are products $v_1 \otimes v_2$ of dominant vectors in the two factors. Combining the results of Section 2 for v_1 and Section 5 for $v_2 = v$ we see that the positive function ω associated to v must be the product of a theta and/or delta functions on $\mathbf{Z} \times \mathbf{T}$ with the exponential of a quadratic form on V. The Hirota bilinear form characterises dominant vectors in terms of the orthogonal complement of the irreducible representation generated by their tensor square. This is normally done in the Bargmann–Segal representation, in which instead of realising V as sections of a line bundle, we realise a vector v locally in a neighbourhood of a dominant vector u by the holomorphic function

$$v(x) = \frac{\langle W(x)u, v\rangle}{\langle W(x)u, u\rangle}$$

(the normalisation ensuring that u is represented by the constant function 1). The automorphism α of Section 2 is implemented by the rotation R of $V \otimes_S V$ given by $RF(x, y) = F(x+y, x-y)$. One then finds a subspace of vectors $p \in V$ such that for all $q \in V$,

$$\langle q \otimes p, R(v \otimes v) \rangle = 0,$$

when v is dominant. In the Bargmann–Segal representation the adjoint of multiplication by a polynomial is a differential operator, so any p which is represented by a polynomial can be written as P^*u for some differential operator P. After suitable choices of p and changes of variable, the action of P on $R(v \otimes v)$ gives the KP hierarchy as in [7].

This particular approach is unsuited to more general loop groups of the form Map(S^1, G) when G is a semi-simple Lie group, because the constant functions are no longer normal, and one can no longer split the finite and infinite-dimensional parts so easily. Nonetheless, it follows readily from [14, Theorem 9.2.4] that there are dominant vectors which are eigenvectors for the span of all the functions $f_n(z) \in z^n \mathfrak{g}_{\mathbb{C}}$ with $n < 0$ and for constant functions in a borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

In other directions our approach provides more information, as we have characterised all dominant vectors and not just those on the orbit of some particular u. Combining the remarks at the end of the last section and the discussion in [3] links such investigations to studies of Riemann surfaces.

7. Generalised dominant vectors

It will be apparent from the discussion of Section 4 that in the case of induced representations such as those of the Euclidean group, it would be useful to deal with functions supported at a single point, since then $\psi \otimes \psi$ would also be concentrated at a single point and therefore in a single double coset in Mackey's tensor product decomposition. To achieve this one wishes to deal with distributions, or in the general context with elements of the dual to some subspace of test functions in the representation space. Maurin [11, Chapter XVII.6], has shown that for any strongly continuous unitary representation of a Lie group G in a separable Hilbert space $\mathcal{H}$ there exists a G-invariant nuclear subspace Φ of the Gårding space of C^∞-vectors such that the centre of the enveloping algebra has a complete set of generalised eigenvectors in Φ'. For an irreducible representation weak measurability is sufficient, and the centre will in any case act as scalars. For the left regular representation Maurin's construction gives the Schwartz space $\mathcal{D}(G)$.

The vectors in the representation space of $\sigma\text{-ind}_{\mathrm{H}}^{G} L$ are functions from G with values in the representation space of L. Once we extend the definition to permit the use of generalised functions we can construct dominant generalised vectors v of the form $v(x) = \delta_{\mathrm{H}}(x)v$, where δ_{H} is a unit measure concentrated on the subgroup H, and v is the dominant generalised vector for L. (The discussion of Section 3 shows that not every dominant generalised vector is of this form.)

We shall not pursue this idea in detail, but will content ourselves with a single example of the finite-dimensional algebra CCR(V). We allow the bilinear form B in the

σ-positive function ω to become singular in certain directions (cf. Section 2). The inequality $B(x,x)B(y,y) \geq s(x,y)^2$ then forces divergence in other directions, and shows that $B(y,y)$ can vanish at most for y in some Lagrangian subspace L of V. The net result, however, is that in the limit we obtain generalised σ-positive functions of the form

$$\omega(x) = \mathrm{e}^{\mathrm{i}\lambda(x)}\delta_{\mathrm{L}}(x),$$

where δ_{L} is a unit measure concentrated on the subspace L, and λ need only be defined on L. This is precisely the form of the σ-positive function obtained for the σ-induced representation $\sigma\text{-ind}_{\mathrm{L}}^{V}\,\lambda$. Such ω also satisfy the sort of identity given earlier since

$$\omega(x+y)\omega(x-y) = \mathrm{e}^{2\mathrm{i}\lambda(x)}\delta_{\mathrm{L}}(x+y)\delta_{\mathrm{L}}(x-y) = \mathrm{e}^{2\mathrm{i}\lambda(x)}\delta_{\mathrm{L}}(x)\delta_{\mathrm{L}}(y).$$

In principle, rather than going via limits one could directly classify the generalised σ-positive functions corresponding to generalised dominant vectors.

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